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Asymptotic (short-wave) equivalence of one-dimensional Schrödinger equations by formal canonical transformations and its generalizations

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Abstract. We define the asymptotic equivalence of systems of two ordinary first-order differential equations with an arbitrary finite number and multiplicities of turning points. The theory is exemplified by the short-wave (semiclassical) equivalence of time-independent one-dimensional Schrödinger equations. We describe the set of all transformation matrices realizing the equivalence; the use of their determinant properties simplifies the calculations needed for applications. For the particular case of Schrödinger equations the transformation matrix can be chosen to be canonical.

1. Introduction

Many phenomena in nuclear, atomic and molecular physics and related branches of science can be described within the framework of a semiclassical (i.e. short-wave) approach, which heavily exploits the fact that the Planck constant $\hbar$ may be effectively considered as a small parameter (e.g. see, Fröman and Fröman 1965, Berry and Mount 1972, Landau and Lifshitz 1977, Child 1980, Maslov and Fedoriuk 1981, Eu 1984, Keller 1985). Therefore, from the mathematical viewpoint, semiclassical mechanics is a particular case of the asymptotic theory of differential equations. As a concrete example, consider the time-independent Schrödinger equation for a system of one degree of freedom:

$$\left(\hbar^2 \frac{d^2}{dx^2} + p(x)\right)\Psi(x) = 0 \tag{1}$$

where $p(x) = 2m(E - V(x))$, m is the mass of the particle, $V(x)$ the potential and E the energy. In physical applications such as the scattering theory one often has to deal with solutions of this equation defined throughout the domain over which the independent variable x ranges. One of the powerful methods of studying the solutions Ψ of equation (1) is to represent them (to an arbitrary order of $\hbar$) in terms of solutions Q and their first derivatives Q' of another equation of the same form,

$$\left(\hbar^2 \frac{d^2}{dz^2} + f(z)\right)Q(z) = 0 \tag{2}$$

which is called a *related* or *comparison* equation and whose solutions are supposed to be known or easier to investigate (e.g. $f(z)$ being a polynomial). This approach, first proposed and developed by Langer (1949), stems from techniques best known as

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the Liouville–Green to mathematicians and JWKB to physicists and can be thought as a rigorous variant of phase integral methods (see Heading 1962, Guillemin and Sternberg 1977). The representation may be looked for in the form

$$\Psi(\varphi(z)) = M(z)Q(z) + \hbar\Gamma(z)Q'(z) \tag{3}$$

where, in accordance with the asymptotic framework, the coefficients M and Γ are in fact convergent or formal series in powers of $\hbar$. Langer's method has been widely disseminated and employed in many works (for a historical survey see McHugh (1971)). A similar approach for systems of equations was devised by Sibuya (1958), Okubo (1961) and Wasow (1963). A lucid exposition of the principal ideas of the method can be found in Langer (1960) and Wasow (1985). An extensive study of an equation of the form of equation (2) with the function $f(z)$ being a polynomial is presented in Sibuya (1975).

It turns out that the choice of a comparison equation (2) and the procedure of constructing the functions φ, M and Γ in equation (3) are mainly determined by the number and the orders of turning (or transition) points of the original equation (1), i.e. points x at which $p(x)=0$. Langer (1949) considered the case of one simple (i.e. of first order) turning point. Later Kazarinoff (1958) and Langer (1959) treated the case of two turning points of first order, and McKelvey (1955) the case of one turning point of second order. The latter case was also studied by Lee (1969) and O'Malley (1970). An interesting situation of two coalescing simple turning points was dealt with by Olver (1975). Nishimoto (1973) and Sibuya (1974) considered the case of a simple turning point of any integer order. In a remarkable paper by Lynn and Keller (1970) the method was extended to functions $p(x)$ in equation (1) having an arbitrary finite number of turning points of any order. This paper also treated systems of two first-order differential equations. Applications of the uniform asymptotic expansions obtained in Lynn and Keller (1970) to eigenvalue problems are exemplified by Anyanwu and Keller (1975) and to stability problems for Hill's equations, by Weinstein and Keller (1987).

For the case of a single simple turning point, Cherry (1949, 1950) devised a somewhat different approach based on transformations of the independent variable. He used a comparison equation as well, but in his construction the asymptotic expansion did not involve the derivative of a solution of that equation. Zauderer (1972) and Rubenfeld and Willner (1977) generalized Cherry's technique to an arbitrary number and structure of turning points. The results of Zauderer, Rubenfeld and Willner are formal, like those of Lynn and Keller. However, for a single turning point of an arbitrary order Willner and Rubenfeld (1976) showed that their expansions are indeed asymptotic. The approach of Willner and Rubenfeld (1976) resembles that of Sibuya (1974).

Langer's comparison equation method was also used on some higher-order equations. Langer (1955) obtained the uniform asymptotic expansion for a certain type of third-order equations, and Langer (1957) and Lin and Rabenstein (1960, 1969), of fourth-order equations. Anyanwu and Keller (1978) extended the construction of Lynn and Keller (1970) to second-order differential equations and systems of two first-order equations in infinite-dimensional Hilbert spaces and applied their results to obtain the asymptotic solution for propagation of a wave in a slowly varying waveguide.

Lynn and Keller (1970), Zauderer (1972), Rubenfeld and Willner (1977) and Anyanwu and Keller (1978) considered *uniform* asymptotic expansions of solutions of a given equation, i.e. expansions valid throughout a whole domain containing several turning points. A closely related problem is that of connecting the Liouville–Green or

JWKB approximations defined separately in different sectors with the vertex at a given turning point or in different turning-point-free subintervals of the range interval of the independent variable. Olver (1977a) found such connection formulae under the hypothesis that the real independent variable ranges in an interval containing a single turning point of any integer order. Similar results in somewhat different situations (e.g. for the complex independent variable) were obtained by Nishimoto (1973), Sibuya (1974) and Leung (1975). Then Olver (1977b) generalized his formulae to the case of an arbitrary finite number of turning points of any orders. These two papers by Olver also contain a historical survey on the connection formulae approach. Leung (1977) applied connection relations to eigenvalue problems. Connection formulae for an arbitrary number of simple turning points in the complex plane were treated in, for example, Evgrafov and Fedoryuk (1966). General connection formulae for equations in the complex plane were derived by Olver (1978).

Uniform methods are also useful in the asymptotic analysis of differential equations involving simultaneously turning points and singular points. Recent research of such equations and examination of related problems (of which the most important ones are resonances and exponential precision asymptotics) are exemplified by Meyer (1980), Hanson and Tier (1981), Tier and Hanson (1981), Wazwaz and Hanson (1986a, b), Hanson and Wazwaz (1988) and Hanson (1990). The uniform and exponential asymptotic technique is applied for studying the Stokes phenomenon in Berry (1989) and Berry and Howls (1990).

In the present paper we use the comparison equation method in Langer's formulation, following basically the procedure for constructing a representation developed by Lynn and Keller (1970), but while in Lynn and Keller (1970) and most related works the authors' attention was concentrated on the choice of the comparison equation (2) with a function $f(z)$ of the prescribed form, we examine mainly the properties of a transformation converting solutions of equation (2) into solutions of equation (1). So, we consider equations (1) and (2) as having '*equal rights*' and, not confining ourselves to any particular form of the functions $p(x)$ and $f(z)$, describe the set of all the formal uniform transformations. If this set is not empty, we call the equations *asymptotically equivalent*. The solutions of asymptotically equivalent equations exhibit similar qualitative behaviour. In fact, we consider not only one-dimensional Schrödinger equations but arbitrary systems of two linear homogeneous first-order differential equations with a small parameter at the derivatives. Note that recently the problem of the *strict equivalence* for differential equations and operators has been tackled by group-theoretical methods in Kamran and Olver (1989a, b, 1990).

For instance, Kamran and Olver (1989b) solved completely the strict equivalence problem for two second-order linear differential operators on the line. However, the asymptotic equivalence with respect to a small parameter and the strict equivalence involving no small parameters have proved to be drastically different. If one froze a value of the small parameter in two asymptotically equivalent equations (e.g. put $\hbar = 1.0546 \times 10^{-34}$ Js) and treated them as individual equations without any parameter, they would not, generally speaking, be strictly equivalent. Moreover, in our case they may well be even *far from* being strictly equivalent, because the small parameter stands at the derivatives. For example, according to Kamran and Olver (1989b), equations (1) and (2) with x, z real and p, f analytic *are strictly equivalent if and only if the* functions $p(x)$ and $f(z)$ are rescaled translates of each other: $f(z) \equiv a^2 p(az+b)$ with $a \neq 0$ and b arbitrary constants. This result shows that strictly equivalent Schrödinger equations are met with very rarely, and almost no information on the asymptotic equivalence can be obtained in the framework of strict equivalence. The solutions of strictly equivalent equations are quantitatively similar.

Our second purpose is to show how very simple matrix-theoretical arguments, such as the properties of the transformation determinant, allow the number of integrations to be halved in the recursive procedure devised in Lynn and Keller (1970), which seems to be of importance for practical applications. The calculation and persistent use of the transformation determinant is the main innovation of our paper. Relevant examples will be given elsewhere.

We treat only the case where all the functions are defined in some domains of the complex plane and are holomorphic, but an analogous theory can be developed for C^∞ functions in intervals of the real line as well.

The paper is organized as follows. After a precise definition of the asymptotic equivalence of systems in section 2, we discuss the relations between leading-order terms of asymptotically equivalent systems, the role of turning points and the independent variable changes in section 3. A description of all the transformation matrices which link the solutions of asymptotically equivalent systems and a detailed presentation of the recursive algorithm for constructing these matrices are given in section 4. In section 5 we specialize the calculations of the previous section to Schrödinger equations with a single simple turning point. Finally, in the appendix we revisit a delicate existence problem for the independent variable change already discussed by Lynn and Keller (1970), Rubenfeld and Willner (1977) and Willner and Mahar (1977).

2. Asymptotically equivalent systems

Henceforth, we will denote the small parameter by ε rather than $\hbar$ and allow the functions p and f in the equations (1) and (2) to be dependent on ε. The analysis will be more transparent if equations (1) and (2) are written as systems:

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}\Psi\\ \Phi\end{pmatrix} = \begin{pmatrix}0 & 1\\ -p(x,\varepsilon) & 0\end{pmatrix}\begin{pmatrix}\Psi\\ \Phi\end{pmatrix} \tag{4}$$

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix}Q\\ R\end{pmatrix} = \begin{pmatrix}0 & 1\\ -f(z,\varepsilon) & 0\end{pmatrix}\begin{pmatrix}Q\\ R\end{pmatrix} \tag{5}$$

where $\Phi = \varepsilon\, \mathrm{d}/\mathrm{d}x\, \Psi$ and $R = \varepsilon\, \mathrm{d}/\mathrm{d}z\, Q$. Note that the systems (4) and (5) are Hamiltonian, to emphasize this we may rewrite them in the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\Psi = \frac{\partial}{\partial\Phi}H \qquad \frac{\mathrm{d}}{\mathrm{d}x}\Phi = -\frac{\partial}{\partial\Psi}H \qquad H = \frac{1}{2\varepsilon}(\Phi^2 + p\Psi^2)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}z}Q = \frac{\partial}{\partial R}K \qquad \frac{\mathrm{d}}{\mathrm{d}z}R = -\frac{\partial}{\partial Q}K \qquad K = \frac{1}{2\varepsilon}(R^2 + fQ^2)$$

respectively.

Nevertheless, instead of the systems (4) and (5) we will consider more general, not necessarily Hamiltonian, systems

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}\Psi\\ \Phi\end{pmatrix} = A(x,\varepsilon)\begin{pmatrix}\Psi\\ \Phi\end{pmatrix} \tag{6}$$

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix}Q\\ R\end{pmatrix} = B(z,\varepsilon)\begin{pmatrix}Q\\ R\end{pmatrix} \tag{7}$$

where A and B have the asymptotic expansions

$$A(x, \varepsilon) \sim \sum_{k=0}^{\infty} A_k(x)\varepsilon^k$$

$$B(z, \varepsilon) \sim \sum_{k=0}^{\infty} B_k(z)\varepsilon^k$$

$A_k(x)$ and $B_k(z)$ being holomorphic matrix-valued functions defined in some connected domains D and G of the complex plane respectively. The elements of A and B will be denoted as

$$A = \begin{pmatrix} \mu & r \\ -p & \nu \end{pmatrix} \qquad B = \begin{pmatrix} \sigma & g \\ -f & \tau \end{pmatrix} \tag{8}$$

with the asymptotic expansions

$$\mu(x, \varepsilon) \sim \sum_{k=0}^{\infty} \mu_k(x)\varepsilon^k$$

$$p(x, \varepsilon) \sim \sum_{k=0}^{\infty} p_k(x)\varepsilon^k \qquad \ldots .$$

In these notations the equations (4) and (5), which are of most importance for physical applications, correspond to

$$\mu_k(x) = \nu_k(x) \equiv 0 \qquad \sigma_k(z) = \tau_k(z) \equiv 0$$

for all $k \geqslant 0$,

$$r_0(x) \equiv 1 \qquad g_0(z) \equiv 1$$

and

$$r_k(x) \equiv 0 \qquad g_k(z) \equiv 0$$

for all $k \geqslant 1$.

Definition 1. The systems (6) and (7) are asymptotically equivalent if there exist a bijective (i.e. one-to-one) holomorphic function $\varphi : G \to D$ and a matrix-valued function $S(z, \varepsilon)$, $z \in G$, with the asymptotic expansion

$$S(z, \varepsilon) \sim \sum_{k=0}^{\infty} S_k(z)\varepsilon^k \qquad \det S_0(z) \neq 0$$

$S_k(z)$ being holomorphic in G, such that the change of variables

$$\begin{pmatrix} \Psi(\varphi(z)) \\ \Phi(\varphi(z)) \end{pmatrix} = S(z, \varepsilon) \begin{pmatrix} Q(z) \\ R(z) \end{pmatrix} \tag{9}$$

transforms the system (7) into the system (6).

This definition is compatible with that given in Wasow (1985).

We call φ and S the transformation function and the transformation matrix respectively and will speak of equivalent systems rather than asymptotically equivalent ones for brevity. Note that the bijective property of φ implies $\mathrm{d}/\mathrm{d}z\, \varphi \neq 0$ everywhere in G (e.g. see, Markushevich 1977).

Our purpose is to find the restrictions which the equivalence imposes on the matrices $A(x, \varepsilon)$ and $B(z, \varepsilon)$, and to describe the class of all transformation matrices $S(z, \varepsilon)$ provided that the transformation function $\varphi(z)$ is given.

Henceforth, a prime will denote differentiating with respect to z. Substituting equation (9) into equations (6) and (7) we obtain the transformation equation

$$\varphi'(z)A(\varphi(z), \varepsilon)S(z, \varepsilon) - S(z, \varepsilon)B(z, \varepsilon) = \varepsilon S'(z, \varepsilon)$$

or, in short notation,

$$\eta AS - SB = \varepsilon S' \tag{10}$$

where we have set

$$\eta(z) = \varphi'(z)$$

for the sake of brevity.

Using the well known matrix-theoretical relation

$$(\det S)' = (\det S)\,\mathrm{tr}(S^{-1}S')$$

valid for any function $S(z)$ such that $\det S(z) \neq 0$ (e.g. see, Arnold (1978a), where this relation is proven in a particular case of Wronskians of systems of linear differential equations) and recalling that $\mathrm{tr}(S^{-1}AS) = \mathrm{tr}\, A$, we obtain from equation (10) that

$$(\det S)' = \det S \frac{\eta\, \mathrm{tr}\, A - \mathrm{tr}\, B}{\varepsilon}. \tag{11}$$

3. Normalization of zeroth-order terms

Consider a system

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}\xi}\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = L(\xi, \varepsilon)\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \tag{12}$$

where ξ is a complex variable and the matrix L has the asymptotic expansion

$$L(\xi, \varepsilon) \sim \sum_{k=0}^{\infty} L_k(\xi)\varepsilon^k$$

with holomorphic coefficients $L_k(\xi)$.

Definition 2. A value ξ_* of the independent variable ξ is a turning point of the system (12) if the two eigenvalues of the matrix $L_0(\xi_*)$ are equal, i.e. $4 \det L_0(\xi_*) = (\mathrm{tr}\, L_0(\xi_*))^2$. The order of a turning point ξ_* is the order of the number ξ_* as a zero of the function $4 \det L_0(\xi) - (\mathrm{tr}\, L_0(\xi))^2$. A turning point ξ_* is non-degenerate if the Jordan structure of $L_0(\xi_*)$ is a single block of order two, and degenerate if $L_0(\xi_*)$ is diagonal.

Example. If $L_0(\xi)$ is of the form

$$L_0(\xi) = \begin{pmatrix} 0 & 1 \\ -q_0(\xi) & 0 \end{pmatrix} \tag{13}$$

then the turning points of the system (12) are the zeros of the function $q_0(\xi)$. All these turning points are non-degenerate.

Lemma. If systems (6) and (7) are equivalent then they have the same number of turning points of the same orders and the same degeneracy types. Moreover, every transformation function $\varphi(z)$ maps the turning points of the system (7) to the corresponding turning points of the system (6). The relations

$$\begin{aligned} &\eta^2 \det A_0 = \det B_0 \qquad \eta \operatorname{tr} A_0 = \operatorname{tr} B_0 \\ &\det A_0 (\operatorname{tr} B_0)^2 = \det B_0 (\operatorname{tr} A_0)^2 \end{aligned} \tag{14}$$

hold, where $\eta = \varphi' = \mathrm{d}/\mathrm{d}z\, \varphi$. Finally, the systems with matrices

$$A(x, \varepsilon) - \tfrac{1}{2} I \operatorname{tr} A_0(x) \qquad \text{and} \qquad B(z, \varepsilon) - \tfrac{1}{2} I \operatorname{tr} B_0(z)$$

are also equivalent with the same $\varphi(z)$ and $S(z, \varepsilon)$ (I being 2×2 unity matrix).

Proof. Equating zeroth-order terms in the transformation equation (10) we obtain

$$\eta A_0 S_0 = S_0 B_0 \qquad \det S_0 \neq 0$$

which immediately implies all the statements of the lemma (the last statement is obvious in view of the fact that

$$\eta(A - \tfrac{1}{2} I \operatorname{tr} A_0) S - S(B - \tfrac{1}{2} I \operatorname{tr} B_0) = \eta A S - S B$$

provided $\eta \operatorname{tr} A_0 = \operatorname{tr} B_0$).

Since $\eta \operatorname{tr} A_0 = \operatorname{tr} B_0$ for equivalent systems, the right-hand side of equation (11) contains no singularity.

Below we will consider only the systems (12) such that $\operatorname{tr} L_0(\xi) \equiv 0$. The lemma above provides us with a justification of this confinement, because $\operatorname{tr}(L_0 - \frac{1}{2} I \operatorname{tr} L_0) \equiv 0$. Another justification stems from the fact that the solutions of the system $\mathrm{d}/\mathrm{d}\xi\, U = \Lambda U$ and $\mathrm{d}/\mathrm{d}\xi\, \tilde{U} = (\Lambda + \lambda I) \tilde{U}$ with a scalar function $\lambda(\xi)$ are linked by a simple relation

$$\tilde{U} = U \exp\left(\int_{\xi_0}^{\xi} \lambda(\xi)\, \mathrm{d}\xi \right).$$

Finally, we are mostly interested in the systems (12) with matrices $L_0(\xi)$ of the form (13) for which $\operatorname{tr} L_0(\xi) \equiv 0$.

Lynn and Keller (1970, section 9) declared that for any system (12) with $\operatorname{tr} L_0(\xi) \equiv 0$ there exists a holomorphic matrix-valued function $V(\xi)$ such that $\det V(\xi) \neq 0$ and

$$V^{-1} L_0 V = \begin{pmatrix} 0 & 1 \\ -q_0(\xi) & 0 \end{pmatrix}$$

provided that all the turning points of the system (12) are non-degenerate. Then the system

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}\xi} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = V^{-1} \left(L V - \varepsilon \frac{\mathrm{d}V}{\mathrm{d}\xi} \right) \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$$

is clearly equivalent to the original one and

$$V^{-1} \left(L V - \varepsilon \frac{\mathrm{d}V}{\mathrm{d}\xi} \right) = \begin{pmatrix} 0 & 1 \\ -q_0(\xi) & 0 \end{pmatrix} + \mathrm{O}(\varepsilon).$$

In view of this and being oriented to the physical applications, we will consider below only the systems (12) with the matrix $L_0(\xi)$ of the form (13).

Degenerate turning points of second-order systems of linear differential equations were extensively studied in Hanson and Russell (1967) and Hanson (1968).

Lynn and Keller (1970, section 9) treated only the case of systems (6) and (7) for which the matrix A has a normalized zeroth-order term

$$A_0(x)=\begin{pmatrix}0 & 1\\ -p_0(x) & 0\end{pmatrix}$$

whereas the matrix B is entirely off diagonal with the right upper element equal to unity, i.e. it is of the form (5):

$$B(z,\varepsilon)=\begin{pmatrix}0 & 1\\ -f(z,\varepsilon) & 0\end{pmatrix}$$

$f(z,\varepsilon)$ being a polynomial in z.

4. Recursive procedure

Consider two systems (6) and (7) with normalized zeroth-order terms in the asymptotic expansions of the matrices $A(x,\varepsilon)$ and $B(z,\varepsilon)$:

$$A_0(x)=\begin{pmatrix}0 & 1\\ -p_0(x) & 0\end{pmatrix}\qquad B_0(z)=\begin{pmatrix}0 & 1\\ -f_0(z) & 0\end{pmatrix}.$$

According to the notation (8) this means $\mu_0(x)=\nu_0(x)\equiv 0$, $\sigma_0(z)=\tau_0(z)\equiv 0$, $r_0(x)\equiv 1$, $g_0(z)\equiv 1$. We will suppose that in the domain D the function p_0 has $n+1\geqslant 1$ zeros $x_0, x_1, \ldots, x_n$ of orders $m_0, m_1, \ldots, m_n$ which are the turning points of the system (6), and in the domain G the function f_0 has $n+1$ zeros $z_0, z_1, \ldots, z_n$ of the same orders $m_0, m_1, \ldots, m_n$ which are the turning points of the system (7). The relations (14) in this case reduce to

$$\eta^2 p_0 = f_0 \tag{15}$$

or, in full notation,

$$(\varphi'(z))^2 p_0(\varphi(z)) = f_0(z).$$

The existence of a one-to-one holomorphic function $\varphi: G\to D$ mapping each z_j into x_j, $0\leqslant j\leqslant n$, and satisfying (15) is a rather complicated problem, and we postpone its discussion to the appendix. Here we only point out that equation (15) implies

$$\int_{x_i}^{x_j}\sqrt{p_0(x)}\,\mathrm{d}x=\int_{z_i}^{z_j}\sqrt{f_0(z)}\,\mathrm{d}z \tag{16}$$

for any corresponding couples x_i, x_j and z_i, z_j of turning points. In the case where the systems (6) and (7) represent one-dimensional Schrödinger equations, the equalities (16) can be recognized as *equivalences of the phase integrals* of importance in the physical applications.

Now, assuming the function $\varphi(z)$ to be chosen, we start looking for the solutions $S(z,\varepsilon)$ of equation (10). Before formulating the main theorem we make two following remarks.

Remark 1. Provided det $S(z_0,\varepsilon)$ is known, det $S(z,\varepsilon)$ for any z in G can be calculated very easily using equation (11).

Remark 2. If $S(z, \varepsilon)$ is a solution of equation (10), so is $W(\varepsilon)S(z, \varepsilon)$ for any power series $W(\varepsilon)$ with a non-zero constant term. The following theorem ensures that all the solutions of equation (10) can be obtained from a particular one in this way.

Henceforth, we will fix a particular determination of the square root $\sqrt{\eta}$.

Theorem. Under the hypotheses on $A_0(x)$, $B_0(z)$ and $\varphi(z)$ stated above, the transformation equation (10) has a formal solution $S(z, \varepsilon)$ if and only if for each $k \geq 1$ the off-diagonal elements p_k, r_k, f_k, g_k of the matrices $A_k(x)$ and $B_k(z)$ (see equation (8)) satisfy the conditions

$$Z_k^1\colon \frac{\mathrm{d}^s}{\mathrm{d}z^s}(\eta^2 p_k - f_k + f_0 r_k - f_0 g_k + T_k)\big|_{z_j} = 0 \tag{17}$$

for all $0 \leq j \leq n$ and $0 \leq s \leq m_j - 2$ and

$$Z_k^2\colon \int_{z_0}^{z_j} (\eta^2 p_k - f_k + f_0 r_k - f_0 g_k + T_k)\frac{\mathrm{d}z}{\sqrt{f_0}} = 0 \tag{18}$$

for all $1 \leq j \leq n$ (totally $m_0 + m_1 + \ldots + m_n - 1$ conditions). Here $T_1 = 0$ and for $k \geq 2$ the T_k is some expression involving the elements of the matrices $A_0, \ldots, A_{k-1}$, $B_0, \ldots, B_{k-1}$. If these conditions are satisfied, then for every sequence $\omega_0, \omega_1, \omega_2, \ldots$ of complex numbers with $\omega_0 \neq 0$ equation (10) has exactly two solutions $S^{(1)}(z, \varepsilon)$ and $S^{(2)}(z, \varepsilon) = -S^{(1)}(z, \varepsilon)$ subject to

$$\det S(z_0, \varepsilon) \sim \sum_{k=0}^{\infty} \omega_k \varepsilon^k. \tag{19}$$

The terms of the asymptotic expansions of these solutions can be computed by a recursive procedure containing a single operation of integrating at each step.

Proof. First of all, introduce the following short notation. For any two sequences $(u_0, u_1, u_2, \ldots)$ and $(v_0, v_1, v_2, \ldots)$ of numbers or functions we will write

$$(uv)_k = \sum_{i=0}^{k} u_{k-i} v_i$$

$$[uv]_k = \sum_{i=0}^{k-1} u_{k-i} v_i$$

$$\{uv\}_k = \sum_{i=0}^{k-2} u_{k-i} v_i$$

$$\langle uv \rangle_k = \sum_{i=1}^{k-1} u_{k-i} v_i$$

(in particular, $[uv]_0 = \{uv\}_0 = \{uv\}_1 = \langle uv \rangle_0 = \langle uv \rangle_1 = 0$).

The elements of S will be denoted as

$$S = \begin{pmatrix} M & \Gamma \\ \Delta & N \end{pmatrix} \tag{20}$$

with the asymptotic expansions

$$M(z,\varepsilon)\sim\sum_{k=0}^{\infty}M_k(z)\varepsilon^k$$

$$\Gamma(z,\varepsilon)\sim\sum_{k=0}^{\infty}\Gamma_k(z)\varepsilon^k. \qquad \ldots$$

Finally, we will use the notation

$$a=\eta\mu-\sigma \qquad b=\eta\nu-\tau$$

$$c=\eta\mu-\tau \qquad d=\eta\nu-\sigma$$

(see equation (8)) and

$$w=a+b=c+d=\eta\ \mathrm{tr}\ A-\mathrm{tr}\ B.$$

In this notation the transformation equation (10) can be rewritten as a sequence of systems $(X_k^1-X_k^4)$:

$$\begin{aligned}
&X_k^1\colon [aM]_k+(f\Gamma)_k+\eta(r\Delta)_k=M'_{k-1}\\
&X_k^2\colon [bN]_k-\eta(p\Gamma)_k-(g\Delta)_k=N'_{k-1}\\
&X_k^3\colon [c\Gamma]_k-(gM)_k+\eta(rN)_k=\Gamma'_{k-1}\\
&X_k^4\colon [d\Delta]_k-\eta(pM)_k+(fN)_k=\Delta'_{k-1}
\end{aligned}\tag{21}$$

$k\geqslant 0$ (here $M_{-1}=N_{-1}=\Gamma_{-1}=\Delta_{-1}=0$) while equation (11) and (19) give

$$\det S(z,\varepsilon)\sim\left(\sum_{k=0}^{\infty}\varepsilon^k\omega_k\right)\exp\left(\sum_{k=0}^{\infty}\varepsilon^k\int_{z_0}^{z}w_{k+1}(z)\,\mathrm{d}z\right)\sim\sum_{k=0}^{\infty}\Omega_k(z)\varepsilon^k \tag{22}$$

where the coefficients Ω_k can be easily expressed in terms of $\omega_0,\ldots,\omega_k, w_1,\ldots,w_{k+1}$. For instance,

$$\Omega_0(z)=\omega_0\exp\left(\int_{z_0}^{z}w_1(z)\,\mathrm{d}z\right)=\omega_0E^2 \tag{23}$$

where we denote

$$E(z)=\exp\left(\tfrac{1}{2}\int_{z_0}^{z}w_1(z)\,\mathrm{d}z\right). \tag{24}$$

In view of equation (15) the sequence of systems (21) is equivalent to the sequence of systems $(Y_k^1-Y_k^4)$:

$$\begin{aligned}
&Y_k^1\colon [aM]_{k+1}+\eta[bN]_{k+1}+[f-\eta^2p,\Gamma]_{k+1}+\eta[r-g,\Delta]_{k+1}=M'_k+\eta N'_k\\
&Y_k^2\colon [c\Gamma]_k-(gM)_k+\eta(rN)_k=\Gamma'_{k-1}\\
&Y_k^3\colon \eta[d\Delta]_{k+1}-f_0[c\Gamma]_{k+1}+[f_0g-\eta^2p,M]_{k+1}+\eta[f-f_0r,N]_{k+1}=\eta\Delta'_k-f_0\Gamma'_k\\
&Y_k^4\colon [bN]_k-\eta(p\Gamma)_k-(g\Delta)_k=N'_{k-1}
\end{aligned}\tag{25}$$

$k\geqslant 0$. Indeed,

$$Y_k^2=X_k^3 \qquad Y_k^4=X_k^2$$

and using equation (15) it can be easily seen that

$$Y_k^1=X_{k+1}^1+\eta X_{k+1}^2 \qquad Y_k^3=\eta X_{k+1}^4-f_0X_{k+1}^3$$

(here the equality $Y_k^1 = X_{k+1}^1 + \eta X_{k+1}^2$ is to be understood in the way that Y_k^1 is obtained multiplying X_{k+1}^2 by η and adding it to X_{k+1}^1, and similarly for all the other equalities). Inversely,

$$X_k^1 = Y_{k-1}^1 - \eta Y_k^4 \qquad X_k^4 = \eta^{-1}(Y_{k-1}^3 + f_0 Y_k^2)$$

(where Y_{-1}^1 and Y_{-1}^3 are to be ignored).

From Y_k^2 and Y_k^4 we can respectively express M_k and Δ_k:

$$M_k = \eta N_k + \zeta_k \qquad \Delta_k = -\frac{f_0}{\eta}\Gamma_k + \rho_k \tag{26}$$

where

$$\begin{aligned} \zeta_k &= [c\Gamma]_k - [gM]_k + \eta[rN]_k - \Gamma'_{k-1} \\ \rho_k &= [bN]_k - [g\Delta]_k - \eta[p\Gamma]_k - N'_{k-1}. \end{aligned} \tag{27}$$

Note that ζ_k and ρ_k involve only the elements of A_l and B_l with $l \leqslant k$ and S_l with $l < k$. In particular, $\zeta_0 = \rho_0 = 0$ and equation (26) for $k = 0$ reduces to

$$M_0 = \eta N_0 \qquad \Delta_0 = -\frac{f_0}{\eta}\Gamma_0. \tag{28}$$

Substituting equation (26) into those for Y_k^1 and Y_k^3 we arrive, after some simple algebra, at the system for N_k and Γ_k:

$$\begin{aligned} 2\eta N'_k &= -\eta' N_k + \eta w_1 N_k - t_1\Gamma_k + C_k \\ 2\frac{f_0}{\eta}\Gamma'_k &= -\left(\frac{f_0}{\eta}\right)'\Gamma_k + \frac{f_0}{\eta} w_1\Gamma_k + t_1 N_k + D_k \end{aligned} \tag{29}$$

where

$$\begin{aligned} C_k &= \{aM\}_{k+1} + \eta\{bN\}_{k+1} + \{f - \eta^2 p, \Gamma\}_{k+1} + \eta\{r - g, \Delta\}_{k+1} + a_1\zeta_k + \eta(r_1 - g_1)\rho_k - \zeta'_k \\ D_k &= -\{d\Delta\}_{k+1} + \eta p_0\{c\Gamma\}_{k+1} + \{f_0 r - f, N\}_{k+1} + \eta\{p - p_0 g, M\}_{k+1} - d_1\rho_k \\ &\quad + \eta(p_1 - p_0 g_1)\zeta_k + \rho'_k \end{aligned} \tag{30}$$

(C_k and D_k involve only the elements of A_l, B_l with $l \leqslant k+1$ and S_l with $l < k$) and we denote

$$t_k = \eta^2 p_k - f_k + f_0 r_k - f_0 g_k. \tag{31}$$

In particular, $C_0 = D_0 = t_0 = 0$.

The determinant equation (22) can be rewritten as a sequence of equations:

$$(MN)_k - (\Gamma\Delta)_k = \Omega_k. \tag{32}$$

After substituting equations (23) and (28) into equation (32) for $k = 0$, the latter equation takes the form

$$\eta N_0^2 + \frac{f_0}{\eta}\Gamma_0^2 = \omega_0 E^2 \tag{33}$$

whereas after substituting equations (26) for $k \geqslant 1$ and (28) into equation (32) for the same k the latter equation takes the form

$$2\eta N_0 N_k + \frac{2f_0}{\eta}\Gamma_0\Gamma_k = \Omega_k + \Xi_k \tag{34}$$

where

$$\Xi_k = \Gamma_0 \rho_k - N_0 \zeta_k + \langle \Gamma \Delta \rangle_k - \langle MN \rangle_k. \tag{35}$$

Ξ_k involves only the elements of A_l, B_l with $l \leqslant k$ and S_l with $l < k$.

Now we can start successively solving the sequence of systems (29).

4.1. The first step: $k = 0$

The system (29) for $k = 0$ is a system of two first-order homogeneous linear differential equations. Every its solution (N_0, Γ_0) satisfies equation (33), which enables us to solve this system explicitly. Its general solution subject to equation (33) is

$$\begin{aligned} N_0(z) &= \frac{E}{\sqrt{\eta}}(\alpha_0 \cos\theta + \beta_0 \sin\theta) \\ \Gamma_0(z) &= E\sqrt{\frac{\eta}{f_0}}(\alpha_0 \sin\theta - \beta_0 \cos\theta) \end{aligned} \tag{36}$$

where α_0 and β_0 are constants such that $\alpha_0^2 + \beta_0^2 = \omega_0$ and

$$\theta(z) = \frac{1}{2}\int_{z_0}^{z} \frac{t_1}{\sqrt{f_0}}\,\mathrm{d}z \tag{37}$$

t_1 being given by equation (31).

As $f_0(z_0) = 0$ and $\theta(z_0) = 0$ whenever $\theta(z)$ is well defined near z_0, a necessary condition for $\Gamma_0(z)$ to be regular at z_0 is $\beta_0 = 0$. So,

$$\alpha_0^2 = \omega_0$$

and

$$\begin{aligned} N_0(z) &= \frac{\alpha_0 E}{\sqrt{\eta}} \cos\theta \\ \Gamma_0(z) &= \alpha_0 E\sqrt{\frac{\eta}{f_0}} \sin\theta. \end{aligned} \tag{38}$$

Moreover, since $f_0(z)$ vanishes at the turning points z_j to the orders m_j, $0 \leqslant j \leqslant n$, for $\Gamma_0(z)$ to be regular at each z_j one has to require that $\theta(z) = \mathrm{O}((z - z_j)^{m_j/2})$ as $z \to z_j$. This is the case if $t_1(z) = \mathrm{O}[(z - z_j)^{m_j - 1}]$ for each $0 \leqslant j \leqslant n$ and $\theta(z_j) = 0$ for each $1 \leqslant j \leqslant n$. Thus, we arrive at the regularity conditions

$$\left.\frac{\mathrm{d}^s t_1}{\mathrm{d}z^s}\right|_{z_j} \qquad 0 \leqslant j \leqslant n \qquad 0 \leqslant s \leqslant m_j - 2 \tag{39}$$

$$\int_{z_0}^{z} \frac{t_1}{\sqrt{f_0}}\,\mathrm{d}z = 0 \qquad 1 \leqslant j \leqslant n \tag{40}$$

which coincide with the conditions Z_1^1 and Z_1^2 (see equations (17) and (18)).

If these conditions are satisfied, the functions $N_0(z)$ and $\Gamma_0(z)$ defined by equation (38) are indeed holomorphic in G. We should like to specifically emphasize the fact that an ambiguity arising from the determination of the square root $\sqrt{f_0(z)}$ disappears provided that t_1 satisfies equations (39) and (40). This follows from the oddness of the sine and the evenness of the cosine.

4.2. *The subsequent steps: $k \geqslant 1$*

The general solution of the system (29) for $k \geqslant 1$ is

$$
\begin{aligned}
N_k(z) &= \frac{E}{\sqrt{\eta}} (\alpha_k \cos\theta + \beta_k \sin\theta) \\
\Gamma_k(z) &= E \sqrt{\frac{\eta}{f_0}} (\alpha_k \sin\theta - \beta_k \cos\theta)
\end{aligned}
\tag{41}
$$

where

$$
\alpha_k(z) = \gamma_k + \tfrac{1}{2} \int_{z_0}^{z} \left(\frac{C_k}{\sqrt{\eta}} \cos\theta + \sqrt{\frac{\eta}{f_0}} D_k \sin\theta \right) \frac{\mathrm{d}z}{E} \tag{42}
$$

$$
\beta_k(z) = \delta_k + \tfrac{1}{2} \int_{z_0}^{z} \left(\frac{C_k}{\sqrt{\eta}} \sin\theta - \sqrt{\frac{\eta}{f_0}} D_k \cos\theta \right) \frac{\mathrm{d}z}{E} \tag{43}
$$

γ_k and δ_k being arbitrary constants, and θ being given by equation (37).

As $f_0(z_0) = 0$ and $\theta(z_0) = 0$, a necessary condition for $\Gamma_k(z)$ to be regular at z_0 is $\beta_k(z_0) = 0$, which is equivalent to

$$
\delta_k = 0 \tag{44}
$$

because $\beta_k(z_0) = \delta_k$ whenever $\beta_k(z)$ is well defined near z_0. Moreover, since $f_0(z) = \mathrm{O}[(z - z_j)^{m_j}]$ as $z \to z_j$ for each turning point z_j, for $\Gamma_k(z)$ to be regular at z_j one has to require that $\beta_k(z) = \mathrm{O}[(z - z_j)^{m_j/2}]$ as $z \to z_j$, $0 \leqslant j \leqslant n$. Thus, we arrive at the regularity conditions

$$
\frac{\mathrm{d}^s}{\mathrm{d}z^s} \left(\sqrt{\frac{f_0}{\eta}} C_k \sin\theta - \sqrt{\eta}\, D_k \cos\theta \right) \frac{1}{E} \bigg|_{z_j} = 0 \qquad 0 \leqslant j \leqslant n \qquad 0 \leqslant s \leqslant m_j - 2 \tag{45}
$$

$$
\int_{z_0}^{z_j} \left(\frac{C_k}{\sqrt{\eta}} \sin\theta - \sqrt{\frac{\eta}{f_0}} D_k \cos\theta \right) \frac{\mathrm{d}z}{E} = 0 \qquad 1 \leqslant j \leqslant n. \tag{46}
$$

If these conditions are satisfied, the functions $N_k(z)$ and $\Gamma_k(z)$ defined by equation (41) in which α_k and β_k are given by equations (42)–(44) are indeed holomorphic in G. In particular, an ambiguity arising from the determination of the square root $\sqrt{f_0(z)}$ disappears.

Suppose that the regularity conditions (45) and (46) are satisfied and one desires to calculate the functions $N_k(z)$ and $\Gamma_k(z)$ according to equations (41)–(44). Each of the equations (42) and (43) contains an integral. Nevertheless, one has to compute the integral in formula (43) for $\beta_k(z)$ only, because $\alpha_k(z)$ can be found very easily using equation (34) provided that Ω_k has been already calculated from equation (22). Indeed, substituting equations (38) and (41) into equation (34) we obtain

$$
\alpha_k = \frac{\Omega_k + \Xi_k}{2\alpha_0 E^2}. \tag{47}
$$

Here Ξ_k is given by equation (35) and Ω_k is defined by equation (22). From the latter equation it is not hard to derive that

$$
\Omega_k = \omega_k E^2 + \dots \tag{48}
$$

where $\dots$ denotes terms involving only $\omega_0, \dots, \omega_{k-1}$. Equations (42), (47) and (48) show how γ_k is uniquely determined by ω_k. Having calculated N_k and Γ_k, one can find M_k and Δ_k by equation (26).

Finally, we can rewrite the expressions (30) for C_k and D_k in the form

$$\begin{aligned} C_k &= a_{k+1}M_0+\eta b_{k+1}N_0+(f_{k+1}-\eta^2 p_{k+1})\Gamma_0+\eta(r_{k+1}-g_{k+1})\Delta_0+\kappa_k \\ D_k &= -d_{k+1}\Delta_0+\eta p_0 c_{k+1}\Gamma_0+(f_0 r_{k+1}-f_{k+1})N_0+\eta(p_{k+1}-p_0 g_{k+1})M_0+\chi_k \end{aligned} \tag{49}$$

κ_k and χ_k involve only the elements of A_l, B_l with $l\leqslant k$ and S_l with $l<k$. Substituting equations (28) and (38) into equation (49) we obtain, after straightforward algebra, that

$$\frac{1}{E}\left(\sqrt{\frac{f_0}{\eta}}\,C_k\sin\theta-\sqrt{\eta}\,D_k\cos\theta\right)=-\alpha_0(t_{k+1}+\Pi_{k+1})$$

where t_{k+1} is given by equation (31) and

$$\Pi_{k+1}=\frac{1}{\alpha_0 E}\left(\sqrt{\eta}\,\chi_k\cos\theta-\sqrt{\frac{f_0}{\eta}}\,\kappa_k\sin\theta\right). \tag{50}$$

So, regularity conditions (45) and (46) gain the form $Z^1_{k+1}-Z^2_{k+1}$ (see equations (17) and (18)) with Π_{k+1} in place of T_{k+1}.

The expression (50) for Π_{k+1} involves the constants $\alpha_0, \omega_1,\ldots,\omega_{k-1}$ determining the matrices $S_0, S_1,\ldots,S_{k-1}$. Nevertheless, if all the regularity conditions $Z^1_l-Z^2_l$ with $1\leqslant l\leqslant k$ are satisfied, regularity conditions (45) and (46) do not depend on these constants. In other words, if a particular collection $(\alpha_0^{(1)},\omega_1^{(1)},\ldots,\omega_{k-1}^{(1)})$ of values of these constants satisfies conditions (45) and (46), so does any other collection $(\alpha_0^{(2)},\omega_1^{(2)},\ldots,\omega_{k-1}^{(2)})$. Indeed, if matrices $S_0^{(1)}(z), S_1^{(1)}(z),\ldots,S_k^{(1)}(z)$ are the holomorphic solutions of the systems $(Y^1_0-Y^4_0), (Y^1_1-Y^4_1),\ldots,(Y^1_k-Y^4_k)$ (see equation (25)) for $\alpha_0=\alpha_0^{(1)}$, $\omega_1=\omega_1^{(1)},\ldots,\omega_{k-1}=\omega_{k-1}^{(1)}$, $\omega_k=\omega_k^*$ with some ω_k^* then the matrices $S_0^{(2)}(z), S_1^{(2)}(z),\ldots,S_k^{(2)}(z)$ defined by

$$\begin{aligned}&\left((\alpha_0^{(1)})^2+\sum_{i=1}^{k-1}\omega_i^{(1)}\varepsilon^i+\omega_k^*\varepsilon^k\right)^{1/2}\sum_{i=0}^{k}\varepsilon^i S_i^{(2)}(z)\\ &\qquad=\left((\alpha_0^{(2)})^2+\sum_{i=1}^{k-1}\omega_i^{(2)}\varepsilon^i+\omega_k^*\varepsilon^k\right)^{1/2}\sum_{i=0}^{k}\varepsilon^i S_i^{(1)}(z)+O(\varepsilon^{k+1})\end{aligned} \tag{51}$$

are the holomorphic solutions of the same systems for $\alpha_0=\alpha_0^{(2)}$, $\omega_1=\omega_1^{(2)},\ldots,\omega_{k-1}=\omega_{k-1}^{(2)}$, $\omega_k=\omega_k^*$ (the square roots in equation (51) are determined so that the square root on the left-hand side is equal to $\alpha_0^{(1)}+O(\varepsilon)$, and the square root on the right-hand side, to $\alpha_0^{(2)}+O(\varepsilon)$). Hence, conditions (45) and (46) are satisfied for the collection $(\alpha_0^{(2)},\omega_1^{(2)},\ldots,\omega_{k-1}^{(2)})$, too. Thus we can define T_{k+1} as Π_{k+1} calculated at $\alpha_0=1$, $\omega_1=\ldots=\omega_{k-1}=0$ and obtain the regularity conditions $Z^1_{k+1}-Z^2_{k+1}$. The proof of the theorem is completed.

5. Application to the case of a single simple turning point

For 2×2 matrices, the Lie algebra $\mathrm{sp}(2,\mathbb{C})$ of complex *infinitesimally symplectic* matrices (i.e. those determining Hamiltonian linear differential equations) is isomorphic to the Lie algebra $\mathrm{sl}(2,\mathbb{C})$ of matrices with trace zero, and correspondingly the Lie group $\mathrm{SP}(2,\mathbb{C})$ of complex symplectic matrices (i.e. those determining canonical linear transformations) is isomorphic to the Lie group $\mathrm{SL}(2,\mathbb{C})$ of matrices with determinant unity (e.g. see, Arnold (1978b, 1988) or Arnold and Givental (1990)). If the matrices A and B in equations (6) and (7) are of trace zero, which is the case, for example, for systems corresponding to one-dimensional Schrödinger equations, then in accordance with equation (11) det S is independent of z and in equation (22) $\Omega_k(z) \equiv \omega_k$. According to the theorem of the previous section we can set $\omega_0 = 1$, $\omega_k = 0$ for $k \geqslant 1$ and look for the transformation matrix $S(z, \varepsilon)$ within the class of symplectic matrices. If the transformation equation (10) is solvable it has exactly two symplectic solutions $S^{(1)}$ and $S^{(2)} = -S^{(1)}$.

Unfortunately, the particular case of systems (4) and (5) corresponds to no simplifications in our procedure. From the regularity conditions Z_k^1 and Z_k^2 (see equations (17) and (18)) for $k \geqslant 2$ together with equations (27), (30), (49) and (50) for $k \geqslant 1$ it is seen that even when $p(x, \varepsilon) = p_0(x)$ does not depend on ε, one has to take into account higher-order terms $f_1(z), f_2(z), \ldots$, of $f(z)$ to achieve the equivalence of the systems.

The only situation where some remarkable simplifications take place is the case of a single simple turning point. In this case there are no regularity conditions to be satisfied, and it becomes possible to set $p(x, \varepsilon) = p_0(x)$, $f(z, \varepsilon) = f_0(z)$, which leads to the following result.

Proposition. If the matrices A and B in equations (6) and (7) have the form

$$A(x, \varepsilon) = A(x) = \begin{pmatrix} 0 & 1 \\ -p_0(x) & 0 \end{pmatrix} \qquad B(z, \varepsilon) = B(z) = \begin{pmatrix} 0 & 1 \\ -f_0(z) & 0 \end{pmatrix}$$

and the domains of independent variables x and z contain single first-order turning points x_0 and z_0 respectively, then the terms $S_k(z)$ of the symplectic transformation matrix $S(z, \varepsilon)$ are diagonal for k even and off diagonal for k odd.

Proof. Under the hypotheses of the proposition,

$$\zeta_k = -\Gamma'_{k-1} \qquad \rho_k = -N'_{k-1}$$

(see equation (27))

$$M_k = \eta N_k - \Gamma'_{k-1} \qquad \Delta_k = -\frac{f_0}{\eta}\Gamma_k - N'_{k-1} \tag{52}$$

(see equation (26))

$$C_k = \Gamma''_{k-1} = \kappa_k \qquad D_k = -N''_{k-1} = \chi_k$$

(see equations (30) and (49)), and the equations (29) take the form

$$2\eta N'_k = -\eta' N_k + \Gamma''_{k-1}$$

$$2\frac{f_0}{\eta}\Gamma'_k = -\left(\frac{f_0}{\eta}\right)'\Gamma_k - N''_{k-1}.$$

Moreover, in this case $E(z) \equiv 1$ (see equation (24)) and $\theta(z) \equiv 0$ (see equation (31) for $k = 1$ and equation (37)). So, according to equations (35), (38), (41)–(44) and (47) we arrive at

$$N_0 = \frac{\alpha_0}{\sqrt{\eta}} \qquad \Gamma_0 = 0 \tag{53}$$

and

$$N_k = \frac{1}{\sqrt{\eta}}\left(\gamma_k + \frac{1}{2}\int_{z_0}^{z} \frac{\Gamma''_{k-1}}{\sqrt{\eta}}\,\mathrm{d}z\right) = \frac{1}{\sqrt{\eta}}\,\frac{\omega_k + N_0\Gamma'_{k-1} + \langle\Gamma\Delta\rangle_k - \langle MN\rangle_k}{2\alpha_0}$$
$$\Gamma_k = -\frac{1}{2}\sqrt{\frac{\eta}{f_0}}\int_{z_0}^{z}\sqrt{\frac{\eta}{f_0}}\,N''_{k-1}\,\mathrm{d}z \tag{54}$$

for $k \geq 1$. The symplectic property of the matrix S implies that $\omega_k = 0$ for $k \geq 1$. Now from equations (52)-(54) it immediately follows by induction that $\Gamma_k = \Delta_k = 0$ for k even and $N_k = M_k = 0$ for k odd. The proposition is proven. Note that $S_0 = \alpha_0 \operatorname{diag}(\sqrt{\eta}, 1/\sqrt{\eta})$. Moreover, as N_k can be found without integration and $\Gamma_k = 0$ for k even, we arrive at the conclusion that to calculate $S_k(z)$ for k even one has to perform no integrations at all. Thus, under the hypotheses of the proposition, the number of integrations to be fulfilled is two times less than in the general case.

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Appendix. On the existence problem for the transformation function

Let G be a connected domain in the complex plane $\mathbb{C}$ and let holomorphic functiond $p_0: \mathbb{C} \to \mathbb{C}$ and $f_0: G \to \mathbb{C}$ have zeros $x_0, x_1, \ldots, x_n$ and $z_0, z_1, \ldots, z_n$ respectively, of the same orders $m_0, m_1, \ldots, m_n$. Suppose that

$$\int_{x_0}^{x_j} \sqrt{p_0(x)}\,\mathrm{d}x = \int_{z_0}^{z_j} \sqrt{f_0(z)}\,\mathrm{d}z \tag{A1}$$

$(1 \leq j \leq n)$, for a particular determination of the square roots along the integration paths. Lynn and Keller (1970, section 2 and appendix) declared that under these hypotheses there always exists a holomorphic function $\varphi: G \to \mathbb{C}$ such that:

(i) $\varphi(z_j) = x_j$ for $0 \leq j \leq n$;

(ii) $\int_{x_0}^{\varphi(z)} \sqrt{p_0(x)}\,\mathrm{d}x = \int_{z_0}^{z} \sqrt{f_0(z)}\,\mathrm{d}z$ for z in G (note that this equality implies equation (15));

(iii) $\varphi'(z) \neq 0$ everywhere in G.

(In fact, Lynn and Keller considered only the case where $p_0(x)$ is a polynomial

$$p_0(x) = c\prod_{j=0}^{n}(x - x_j)^{m_j} \qquad c = \text{const}$$

of degree $m_0 + m_1 + \ldots + m_n$.) In reality, this statement is not correct, as the following counterexample shows.

In this counterexample p_0 and f_0 are polynomials of degree six, each having three double roots. Namely,

$$p_0(x) = x^2(x-a)^2(x-b)^2$$

where a and b are arbitrary real positive constants such that $a<b<2a$, and

$$f_0(z)=(z+v)^2z^2(z-u)^2$$

where real positive constants u and v are determined by the system

$$u^3(u+2v)=b^3(2a-b) \qquad v^3(v+2u)=a^3(2b-a)$$

(it is not hard to prove that this system has a unique solution (u, v) with $u>0$, $v>0$).

Having determined $\sqrt{p_0(x)}$ and $\sqrt{f_0(z)}$ as

$$\sqrt{p_0(x)}=x(x-a)(x-b)$$

and

$$\sqrt{f_0(z)}=-(z+v)z(z-u)$$

we obtain

$$J_1=\int_0^b \sqrt{p_0(x)}\,\mathrm{d}x=\int_0^u \sqrt{f_0(z)}\,\mathrm{d}z=\frac{b^3(2a-b)}{12}$$

and

$$J_2=\int_0^a \sqrt{p_0(x)}\,\mathrm{d}x=\int_0^{-v} \sqrt{f_0(z)}\,\mathrm{d}z=\frac{a^3(2b-a)}{12}.$$

Thus, all the hypotheses of the statement cited above are satisfied with $n=2$, $m_0=m_1=m_2=2$, $x_0=0$, $x_1=b$, $x_2=a$, $z_0=0$, $z_1=u$ and $z_2=-v$. As G one can take an arbitrarily small complex neighbourhood of the segment $-v\leqslant z\leqslant u$ of the real axis. On the other hand, there exists no holomorphic function $\varphi:G\to\mathbb{C}$ satisfying the conditions (i)-(iii) above. Indeed, let

$$P(x)=\int_{x_0}^x \sqrt{p_0(x)}\,\mathrm{d}x=\frac{x^4}{4}-\frac{(a+b)x^3}{3}+\frac{abx^2}{2}$$

and

$$F(z)=\int_{z_0}^z \sqrt{f_0(z)}\,\mathrm{d}z=-\frac{z^4}{4}+\frac{(u-v)z^3}{3}+\frac{uvz^2}{2}.$$

Suppose that a continuous function $\varphi:G\to\mathbb{C}$ satisfies $\varphi(0)=0$ and $P(\varphi(z))=F(z)$. If a real z increases from 0 to u value $F(z)$ monotonously increases from 0 to J_1. The polynomial $P(x)-J_1$ has the double root b and two simple roots $x^{(1)}<0$ and $0<x^{(2)}<a$. In the interval $0<F\leqslant J_1$ the equation for φ

$$P(\varphi)=F$$

has exactly two solutions $\varphi^{(1)}(F)<\varphi^{(2)}(F)$ which depend on F continuously and tend to zero as $F\to 0$. For these solutions $\varphi^{(1)}(J_1)=x^{(1)}$ and $\varphi^{(2)}(J_1)=x^{(2)}$. So, either $\varphi(u)=x^{(1)}$ or $\varphi(u)=x^{(2)}$ and the function $\varphi(z)$ cannot satisfy the condition $\varphi(u)=b$.

From this counterexample one can also easily understand at which point the argument of Lynn and Keller fails. Indeed, let the equalities (55) hold, $\varphi(z_0)=x_0$ and a function $\varphi(z)$ satisfy the requirements (ii) and (iii) above. Then the equalities (A1) will obviously remain valid if one replaces x_j by $\varphi(z_j)$, $1\leqslant j\leqslant n$. But, despite the reasoning by Lynn and Keller, this does *not* imply that $\varphi(z_j)=x_j$.

An analogous counterexample is provided by polynomials

$$p_0(x)=x^2(x-a)^2(b-x)$$

and

$$f_0(z) = (z+v)^2 z^2 (u-z)$$

of degree five, where a and b are arbitrary real positive constants subject to $a < b < 7a/4$, while real positive constants u and $v > 4u/3$ are determined by the system

$$u^5(4u+7v)^2 = b^5(7a-4b)^2 \qquad (u+v)^5(3v-4u)^2 = (b-a)^5(3a+4b)^2$$

(it is not hard to prove that this system has a unique solution (u, v) with $u > 0$ and $v > 4u/3$).

These counterexamples show that the existence problem for the change $x = \varphi(z)$ of the independent variable is rather delicate. The search for sufficient conditions (presumably in terms of the configuration of the Stokes lines; see Heading 1962, Evgrafov and Fedoryuk 1966, Wasow 1970, 1985, Sibuya 1975) for the existence of φ is beyond the present study.

Note, however, that the failure of the phase integral equivalence (16) to ensure the existence of a transformation function is an essentially complex effect. In the case of infinitely differentiable functions in intervals of the real line the equalities

$$\int_{x_j}^{x_{j+1}} \sqrt{|p_0(x)|}\, dx = \int_{z_j}^{z_{j+1}} \sqrt{|f_0(z)|}\, dz$$

$(0 \leq j \leq n-1)$ under the hypothesis

$$x_0 < x_1 < \ldots x_n \qquad \text{and} \qquad z_0 < z_1 < \ldots z_n$$

(which makes no sense in the complex case) guarantee the existence of the desired function φ. (For details and the proof see Rubenfeld and Willner (1977).) The possibility to achieve the phase integral equivalences in the real case with the function $p_0(x)$ being a polynomial is shown by Willner and Mahar (1977).

Note added. After this paper had been submitted important results by Anyanwu (1988a, b, c) came to our attention. Anyanwu (1988a) has constructed the formal uniform asymptotic theory for differential equations involving simultaneously arbitrary finite numbers of turning points and singular points of any orders. His approach is based on the comparison equation method in Langer's spirit and is entirely parallel to the theory by Lynn and Keller (1970) (in contrast to the technique of Wazwaz and Hanson (1986a, b) which has also been developed for the case of simultaneous presence of zeroes and poles). Similarly to Lynn and Keller (1970), Anyanwu (1988a) considers second-order equations and systems of two first-order equations as well. Moreover, in Anyanwu's paper, there is a crucial lemma concerning the existence of the independent variable change analogous to that by Lynn and Keller. As in Lynn and Keller (1970), the proof of this lemma in Anyanwu (1988a) is given in the appendix and contains the same inaccuracy (see the appendix of present paper).

Anyanwu (1988b) specializes the general expansion of Anyanwu (1988a) for the cases (-2), (-1), (1, -2), (1, -1), (1, 1, -2), (2, -2), and (-1, -1) where numbers 1, 2, -1, and -2 denote a simple turning point, a double turning point, a simple singular point and a double singular point, respectively. Note that the cases (-1) and (-2) were treated in Anyanwu (1982).

In Anyanwu (1988c), expansions found in Anyanwu (1988a, b) are used to solve boundary and eigenvalue problems for second-order differential equations with singular and turning points.

We believe that the equivalence approach with transformation determinant arguments proposed in the present paper can be carried over *mutatis mutandis* to equations with turning and singular points, also.

The theory devised in Anyanwu (1988a, b, c) is applied to obtain uniform approximations for the natural modes and frequencies of some acoustical resonators in Anyanwu and Nwoke (1988).

Among numerous physical applications of uniform asymptotic expansions by Lynn and Keller (1970), we mention here the eigenvalue problems for inhomogeneous dielectric (Arnold 1980a, b) and clad planar (Arnold 1980c, d) waveguides, solving the one-dimensional Schrödinger equation for scattering of a particle by a potential barrier and for bound states of a potential well (Keller 1986), and calculating the change in the action adiabatic invariant for a harmonic oscillator with a slowly varying frequency (Keller and Mu 1991). All these problems can be reduced to equations with two simple turning points.

For a very recent survey on the asymptotic theory for one-dimensional Schrödinger equations, see Slavyanov (1991).

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